Functions of several variables I - Introduction

Introductory example

We can measure the rate of food uptake of a single individual as a function of temperature. We will probably find some optimal temperature T_{opt}, where the uptake rate is highest. At lower temperatures, it is too cold, at higher temperatures, it is too hot for the organism to function properly. If we denote the rate by r and temperature by T then we might try to model this situation with the function

$$r(T) = r_{\max} \exp(-(T - T_{\mathrm{opt}})^2).$$

We can also measure the uptake rate at a constant temperature but in the presence of other individuals. Typically, we see the uptake rate decrease in the presence of others due to competition for food. We have seen such functions before, for example

$$r(N) = r_{\max} \frac{N}{1+N},$$

where N is the number of individuals around. Now we want to vary T and N independently. We could simply multiply the two expressions and get

$$r(N,T) = r_{\max} \frac{N}{1+N} \exp(-(T - T_{\mathrm{opt}})^2).$$

This function now depends on the two variables T and N. While it is easy to plot the two functions of a single variable above, it is much harder to get a good impression of the function of two variables, see Figure 5. Not only is it more difficult to visualize functions of two and more variables, it is also harder to analyze them. The goal of this chapter is to define concepts such as level sets and derivatives for these functions.

Definition: The set $\mathbb{R}^n$ is the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ where all x_i are real numbers. So $\mathbb{R}^1 = \mathbb{R}$, the real numbers; $\mathbb{R}^2$ is the set of points (x_1, x_2) in the plane; $\mathbb{R}^3$ are the points in space. We also use the notation (x, y) and (x, y, z) for points in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

A real-valued function on some subset $D \subset \mathbb{R}^n$ is a function $f\colon D \to \mathbb{R}$ that assigns a real number to each element in D. The set D is called the domain of definition of f.

The graph of a function $f\colon D \to \mathbb{R}$ of two variables is the set

$$G = \{(x, y, z) \in \mathbb{R}^3 \ : \ (x, y) \in D, z = f(x, y)\}.$$

In particular, the graph is a subset of three-dimensional space, and as such, it is not so easy to visualize. Graphs of functions of more variables are defined analogously, but as they are subsets of spaces of dimension 4 and higher, they cannot be visualized in a similar manner.

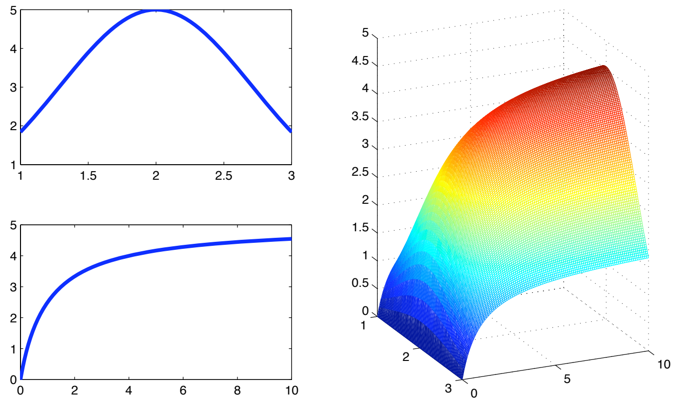


Figure 5: Uptake rate as a function of temperature alone (top left), as a function of population density alone (bottom left), and as a function of both independent variables (right).

Examples

1. $n = 2, D = \mathbb{R}^2, f(x_1, x_2) = x_1^2 + x_2^2$. Then for example

$$f(0,0) = 0, \quad f(1,0) = 1, \quad f(1,1) = 2, f(2,4) = 20.$$

If we fix $x_2 = 0$ the we have a function of a single variable $f(x_1, 0) = x_1^2$. Similarly, if we fix x_1. For a visualization of this function, see Figure 6.

2. $n = 2, D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}, f(x,y) = x + y$. See Figure 6.

3. Find the largest possible domain of the function $f(x,y) = \sqrt{y^2 - x}$. Answer: The square root is real only if $y^2 - x \geq 0$, i.e., $y^2 \geq x$. This gives two conditions:

$$D = \{(x,y) \in \mathbb{R}^2 \; : \; y \geq \sqrt{x} \text{ or } y < -\sqrt{x}\}.$$

See Figure 7 for the domain and the graph of the function.

For functions of two variables, there is a simple way to visualize their behavior. The idea is the same as in topographic maps where the contour lines indicate points of equal altitute.

Definition: The level set, L_c, or contour line of a function $f(x,y)$ is the set of all points $(x,y) \in D$ where f has a given value c, i.e.,

$$L_c = \{(x,y) \in D \; : \; f(x,y) = c\}.$$

Examples, revisited

1. $f(x,y) = x^2 + y^2$. Pick some value c. Then

$$f(x,y) = c \Leftrightarrow x^2 + y^2 = c \Leftrightarrow y = \pm\sqrt{c - x^2}.$$

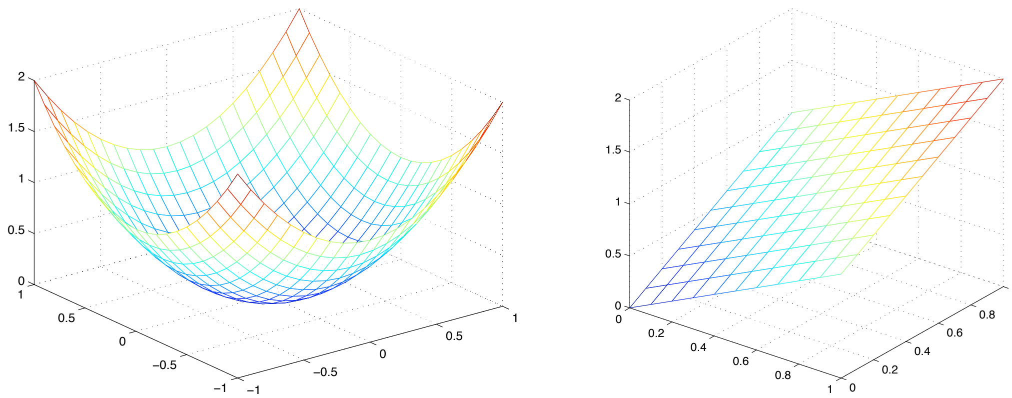


Figure 6: Left: The graph of the function $f(x_1, x_2) = x_1^2 + x_2^2$. Right: The graph of the function $f(x, y) = x + y$.

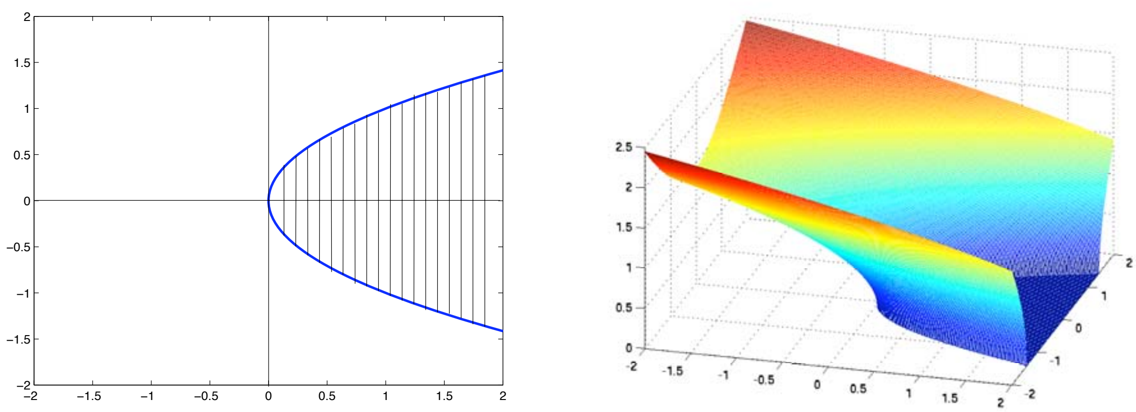
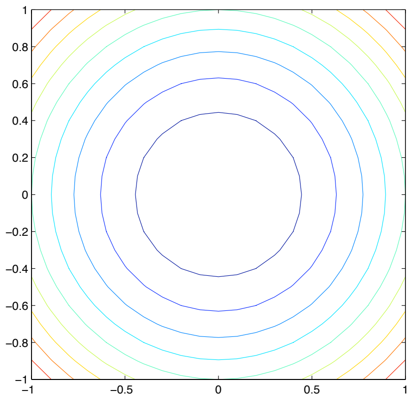


Figure 7: Left: The white area is the maximal domain of definition of the function $f(x, y) = \sqrt{y^2 - x}$. In the shaded area, the square root is not real. Right: The graph of the function.



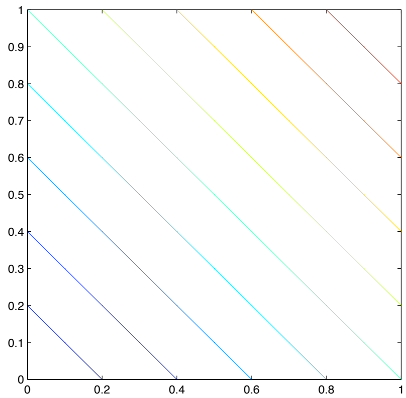


Figure 8: Left: Level sets of the function $f(x_1, x_2) = x_1^2 + x_2^2$. Right: Level sets of the function $f(x, y) = x + y$.

Hence, the level sets for $c > 0$ are concentric circles around the origin. Level sets for $c < 0$ are empty, see Figure 8.

2. $f(x, y) = x + y$. Again, we pick some value c. Then

$$f(x, y) = c \Leftrightarrow x + y = c \Leftrightarrow y = c - x.$$

Hence, the level sets are straight lines with slope -1, see Figure 8.

3. $f(x, y) = \sqrt{y^2 - x}$. Fix c. Then

$$f(x, y) = c \Leftrightarrow y^2 - x = c^2 \Leftrightarrow y = \pm\sqrt{x + c^2}.$$

Hence, the level sets look like the square root function, but they are shifted and flipped, see Figure 9.

We would now like to carry over all the concepts we know from functions of a single variable to functions of several variables, i.e., limits, continuity, derivatives, integrals. All this material is typically taught in a course on "multivariable calculus". It turns out that the general case is much more tricky than the case of a single variable, so much so that we will only cover a small fraction in this course. The following example about continuity shows how things can go "wrong"

Example on limits and continuity

Define the function

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad \text{on} \quad D = \mathbb{R}^2 \setminus \{0\}.$$

What is the limit of f as (x, y) approach the origin $(0, 0)$? Well, it depends on *how* one approaches the origin.

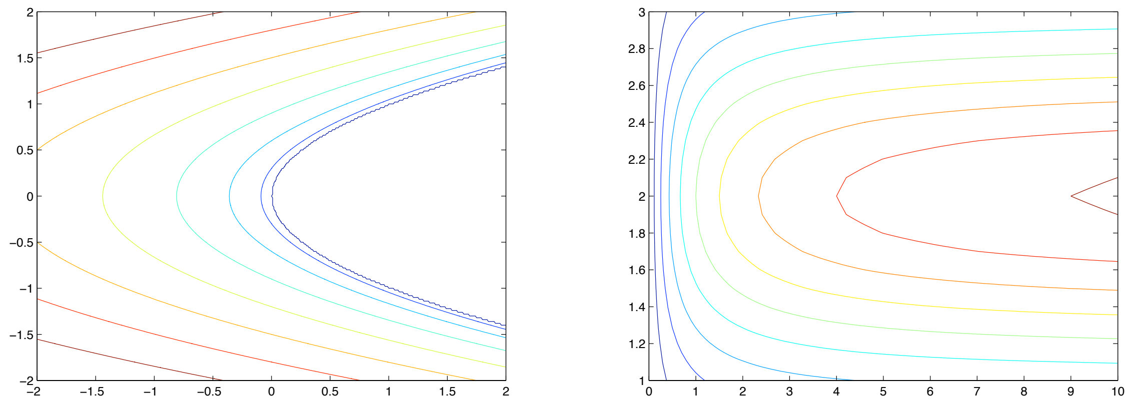


Figure 9: Left: Level sets of the function $f(x, y) = \sqrt{y^2 - x}$. Right: Level sets of the function from the introductory example: $r(N, T) = 5\frac{N}{1+N}\exp(-(T-2)^2)$.

1. At first, fix $y = 0$ and consider the function of a single variable

$$g(x) = f(x, 0) = \frac{x^2 - y^2}{x^2 + y^2}|_{y=0} = \frac{x^2}{x^2} = 1.$$

   Hence, in the limit as $x \to 0$ we get one.

2. Next, fix $x = 0$ and consider the function of a single variable

$$h(y) = f(0, y) = \frac{x^2 - y^2}{x^2 + y^2}|_{x=0} = \frac{-y^2}{y^2} = -1.$$

   Hence, in the limit as $y \to 0$ we get minus one.

3. More generally, pick some $m \neq 0$ and choose the straight line $(x, y) = (x, mx)$ on which to approach the origin. Then in the limit

$$\lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1 - m}{1 + m}.$$

   Hence, the value of the function is constant along such lines, and the limit is different for all the possible straight lines on which we approach the origin. See Figure 10 for a plot of this "weird" function. (Note that we have just computed contour sets for this function.)

Important Observation: We can always obtain functions of a single variable from a function of several variables by holding all but one of the variables constant, as we have done in a few examples above.

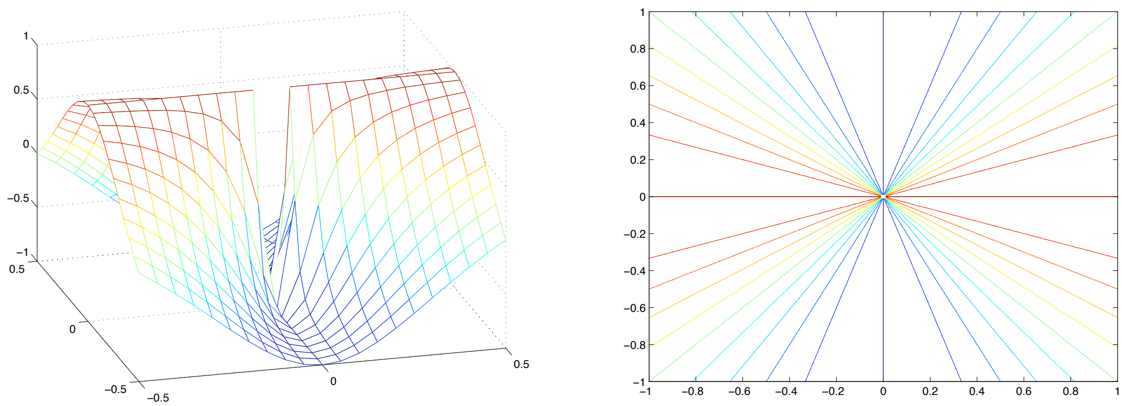


Figure 10: Left: Graph of the function $f(x,y) = (x^2 - y^2)/(x^2 + y^2)$. Right: Level sets of the same function.

Functions of several variables II - Partial derivatives

When we do experiments with several control variables, for example temperature and population density as in the previous section, then we should vary only one of the conditions at a time to see ho the response of the system changes with respect to that particular variable.

This idea is precisely the idea behind the mathematical concept of partial derivative. We fix all the variables but one, we are left with a function of a single variable, and we know how to differentiate that.

Definition: Let $f(x_1, \ldots x_n)$ be a real-valued function of n variables. Then the *partial derivative* with respect to x_k is defined as

$$\frac{\partial f}{\partial x_k} = \lim_{h \to 0} \frac{1}{h}[f(x_1, \ldots, x_k + h, x_{k+1}, \ldots x_n) - f(x_1, \ldots, x_k, \ldots x_n).$$

Note: this is the same definition as for a single variable with all the other variables considered fixed pr parameters. Note also the different notation, the "curly" ∂ versus the straight d.

Examples

1. $f(x,y) = x^2 y$. To find the partial derivative with respect to x, let us fix y. Then we differentiate with respect to x and get.

$$\frac{\partial f}{\partial x} = 2xy.$$

Vise versa, if we fix x and differentiate with respect to y we get

$$\frac{\partial f}{\partial y} = x^2.$$

2. $f(x, y) = ye^{xy}$. Then

$$\frac{\partial f}{\partial x} = y^2 e^{xy}, \quad \frac{\partial f}{\partial y} = e^{xy} + yxe^{xy} = (1 + xy)e^{xy}.$$

3. $f(x, y) = \frac{\sin(xy)}{x^2+1}$. Then

$$\frac{\partial f}{\partial x} = \frac{y\cos(xy)(x^2 + 1) - 2x\sin(xy)}{(x^2 + 1)^2}, \quad \frac{\partial f}{\partial y} = \frac{x\cos(xy)}{x^2 + 1}.$$

4. $f(x, y, z) = e^{xy^2}(1 + z^2)$.

$$\frac{\partial f}{\partial x} = y^2 e^{xy^2}(1 + z^2), \quad \frac{\partial f}{\partial y} = 2xye^{xy^2}(1 + z^2), \quad \frac{\partial f}{\partial z} = 2ze^{xy^2}.$$

Geometric interpretation of partial derivatives

Recall that for a function of a single variable, $g(x)$, the derivative, $g'(x)$, gives the slops of the tangent line at a given point. In fact, the tangent line at a point $(x_0, y_0) = (x_0, g(x_0))$ can be written as

$$y - y_0 = g'(x_0)(x - x_0).$$

Now, for functions of two (or more) variables, the situation is very similar since we obtained partial derivatives from fixing all but one variables and considering essentially a function of a single variable. Therefore, given a function $f(x, y)$ and a point (x_0, y_0) we have $z_0 = f(x_0, y_0)$. Then we can define two functions, namely

$$g_1(x) = f(x, y_0), \qquad g_2(y) = f(x_0, y).$$

Then the partial derivatives $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ are the slopes of the tangent line at the point (x_0, y_0, z_0) to the curves

$$z = f(x, y_0), \qquad \text{and} \qquad z = f(x_0, y),$$

respectively, see Figure 11

In the case of a single variable, we know how to find the *tangent line* to the graph of a function at a point. In the case of two variables, we would like to find the *tangent plane* to the graph of the function, i.e., to the surface in space. This tangent plane necessarily contains the two tangent lines, whose slope we just calculated. Therefore, we can write down the equation of the tangent plane easily.

Fact: Let $f(x, y)$ be a real-valued function of two variables. If the tangent plane to the graph of f at the point $(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ exists, then it is given by the equation

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

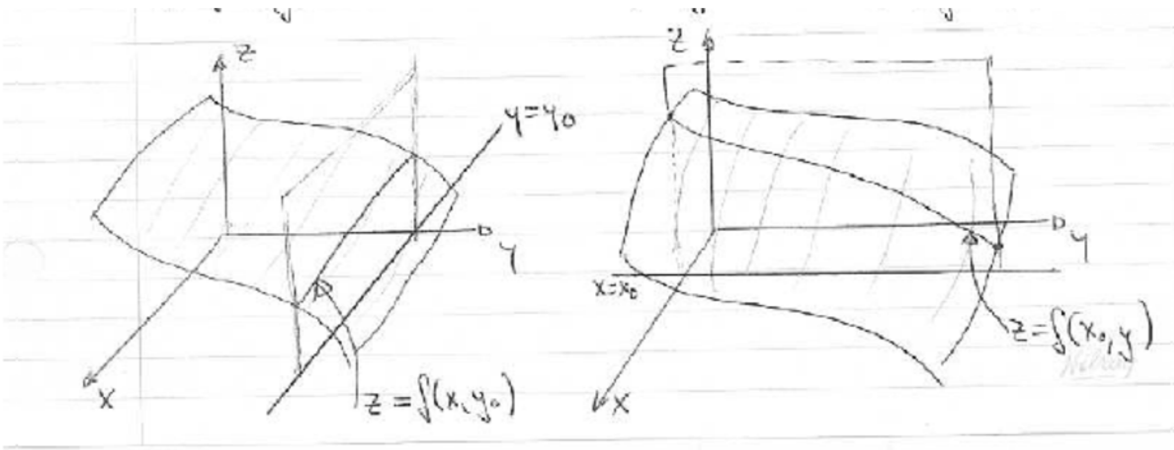


Figure 11: Two one-dimensional curves to which the partial derivatives give the slope.

Examples

1. $f(x, y) = x^2 + y^2$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y.$$

For $(x_0, y_0) = (0, 0)$, we have $f(0, 0) = 0$ and hence the equation of the tangent plane is

$$z - 0 = 0(x - 0) + 0(y - 0), \quad \text{or} \quad z = 0.$$

Hence, the tangent plane is simply the x-y-plane.
At $(x_0, y_0) = (1, 0)$ we have $f(1, 0) = 1$ and hence the equation of the tangent plane there is

$$z - 1 = 2(x - 1) + 0(y - 0), \quad \text{or} \quad z = 2x - 1.$$

At $(x_0, y_0) = (1, 1)$ we have $f(1, 1) = 2$ and hence the equation of the tangent plane there is

$$z - 2 = 2(x - 1) + 2(y - 1), \quad \text{or} \quad z = 2x + 2y - 2.$$

See Figure 12 for a plot of the function and the tangent plane at $(1, 1, 2)$.

2. $f(x, y) = x + y$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = 1.$$

Hence, the tangent plane at (x_0, y_0, z_0) is

$$z - z_0 = x - x_0 + y - y_0 \quad \text{or} \quad z = x + y,$$

since $z_0 = x_0 + y_0$. Hence the tangent plane is exactly the same as the function, which is a plane. We know that the same is true for functions of a single variable, where the tangent line too a linear function is the function itself.

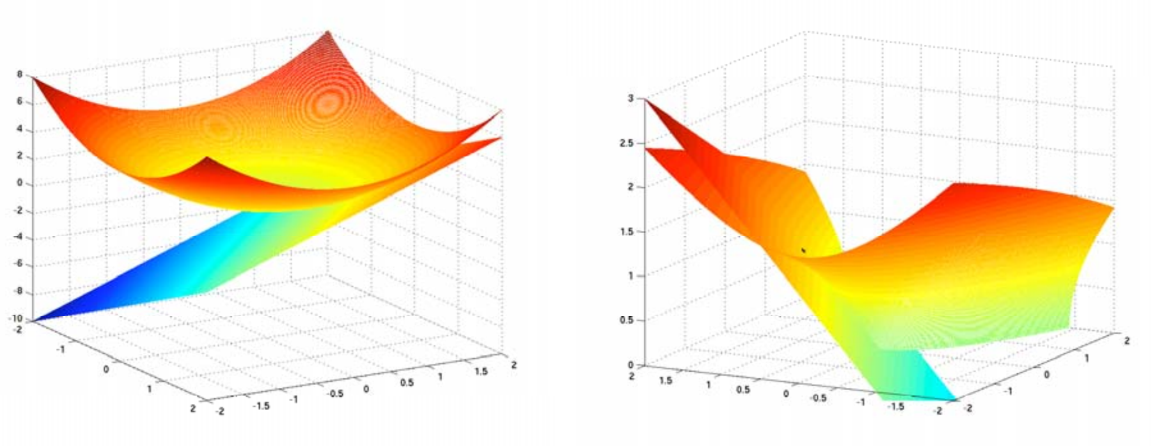


Figure 12: Left: Graph of the function $f(x,y)=x^2+y^2$ and the tangent plane at the point $(1,1,2)$. Right: Graph of the function $f(x,y)=\sqrt{y^2-x}$ and the tangent plane at the point $(0,1,1)$.

3. $f(x,y)=\sqrt{y^2-x}$. The partial derivatives are

$$\frac{\partial f}{\partial x}=-\frac{1}{2\sqrt{y^2-x}},\quad \frac{\partial f}{\partial y}=\frac{y}{\sqrt{y^2-x}}.$$

Then the general equation of the tangent plane is

$$z-z_0=-\frac{1}{2\sqrt{y_0^2-x_0}}(x-x_0)+\frac{y_0}{\sqrt{y_0^2-x_0}}(y-y_0).$$

For example, at $(0,1,1)$ we get

$$z-1=-\frac{1}{2}(x-0)+1(y-1)\quad \text{or}\quad z=-\frac{1}{2}x+y,$$

see Figure 12 for a plot of the function and the tangent plane at $(0,1,1)$.

Linear approximation

Again, we start by recalling the case of a single variable with a function $g(x)$. The tangent line at x_0 gives the linear approximation of the function near that point as

$$g(x)\approx g(x_0)+g'(x_0)(x-x_0),$$

which is also known as the first term in the Taylor series approximation. Now let's do the same thing for a function of two variables, $f(x,y)$ and use the tangent plane to approximate the function locally.

Definition: The linear approximation of a function $f(x,y)$ near a point (x_0, y_0) is given by

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0),$$

provided the function is differentiable.

Examples

1. Find the linear approximation of $f(x,y) = x^2y + 2xe^y$ at the point $(2,0)$.
   First find the partial derivatives

   $$\frac{\partial f}{\partial x} = 2xy + 2e^y, \quad \frac{\partial f}{\partial y} = x^2 + 2xe^y.$$

   Evaluate at the point $(2,0)$ to get

   $$\frac{\partial f}{\partial x} = 2, \quad \frac{\partial f}{\partial y} = 8.$$

   Then with $f(2,0) = 4$ we get

   $$f(x,y) \approx 4 + 2(x-2) + 8(y-0) = 2x + 8y.$$

2. Find the linear approximation of $f(x,y) = \ln(x - 2y^2)$ at the point $(3,1)$.
   First find the partial derivatives

   $$\frac{\partial f}{\partial x} = \frac{1}{x-2y^2}, \quad \frac{\partial f}{\partial y} = \frac{-4y}{x-2y^2}.$$

   Evaluate at the point $(3,1)$ to get

   $$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = -4.$$

   Then with $f(3,1) = \ln(1) = 0$ we get

   $$f(x,y) \approx 0 + 1(x-3) - 4(y-1) = x - 4y + 1.$$

Definition and notation: If $f(x,y)$ is a function of two variables, then the gradient of f is the row-vector

$$\mathrm{grad}(f) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right].$$

The linear approximation can then be written as

$$f(x,y) \approx f(x_0,y_0) + \mathrm{grad}(f)\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}.$$

Functions of several variables III - Vector-valued functions

Motivation

We want to be able to describe several quantities that are all dependent on the same variables. Going back to section 5.5 in the textbook, for example, we described a predator-prey system with prey b and predator p as

$$\begin{aligned}\frac{db}{dt} &= rb - cbp,\\ \frac{dp}{dt} &= \gamma cbp - mp,\end{aligned}$$

with parameters r, c, γ, m. Hence, the growth rate of the prey and the predator both depend on the densities of prey and predator. We can write the right hand side of the above system as a single, vector-valued function of the two variables b, p as follows:

$$F(b,p) = \begin{bmatrix} rb - cbp \\ \gamma cbp - mp \end{bmatrix}.$$

Definition: A vector-valued function F of the variables $x_1, \ldots x_n$ is a function of the form

$$F(x_1, \ldots x_n) = \begin{bmatrix} f_1(x_1, \ldots x_n) \\ f_2(x_1, \ldots x_n) \\ \vdots \\ f_k(x_1, \ldots x_n) \end{bmatrix},$$

where all the functions $f_j(x_1, \ldots x_n)$ are real-valued functions.

Examples

1. The following is a function of two variables that produces a vector of length 2:

$$F(x,y) = \begin{bmatrix} 3x^2 - 2y \\ x^3 \sin(y) \end{bmatrix}.$$

   For every point (x, y) that we put in, we obtain a 2×1-vector. For example

$$F(1,0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad F(2,\pi) = \begin{bmatrix} 12 - 2\pi \\ 0 \end{bmatrix}, \quad F(1,\pi/2) = \begin{bmatrix} 3 - \pi \\ 1 \end{bmatrix}.$$

2. The right-hand side of Newton's law of cooling

$$\begin{aligned}\frac{dH}{dt} &= \alpha(A - H),\\ \frac{dA}{dt} &= \beta(H - A),\end{aligned}$$

see chapter 5.5 in the book, can be written as a vector-valued function

$$F(H,A) = \begin{bmatrix} \alpha(A-H) \\ \beta(H-A) \end{bmatrix}.$$

3. The equation of two competing populations, a, b

$$\begin{aligned} \frac{da}{dt} &= ra(1-a-\eta b), \\ \frac{db}{dt} &= sb(1-b-\mu a), \end{aligned}$$

see chapter 5.5 in the book, can be written as a vector-valued function

$$F(a,b) = \begin{bmatrix} ra(1-a-\eta b) \\ sb(1-b-\mu a) \end{bmatrix}.$$

Linear approximation and Jacobi matrix

Since we will be dealing with differential equations of two variables only in this course, we only consider the case of two variables and two equations, or equivalently, vector-valued functions with two components, i.e.,

$$F(x,y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}.$$

From the previous section, we know how to find the linear approximation for each of the two functions, $f(x,y)$ and $g(x,y)$, namely

$$f(x,y) \approx f(x_0,y_0) + \text{grad}(f) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix},$$

for f and

$$g(x,y) \approx g(x_0,y_0) + \text{grad}(g) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix},$$

for g, respectively. Now we put these two linear approximations together to obtain the linear approximation for $F(x,y)$.

$$F(x,y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \approx \begin{bmatrix} f(x_0,y_0) + \text{grad}(f) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \\ g(x_0,y_0) + \text{grad}(g) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} f(x_0,y_)) \\ g(x_0,y_0) \end{bmatrix} + \begin{bmatrix} \text{grad}(f) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \\ \text{grad}(g) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \end{bmatrix} = F(x_0,y_0) + \begin{bmatrix} \text{grad}(f) \\ \text{grad}(g) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}.$$

Definition and notation: The matrix of partial derivatives

$$J(x,y)=\begin{bmatrix}\text{grad}(f)\\ \text{grad}(g)\end{bmatrix}=\begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}$$

is called the *Jacobi matrix* of the function $F(x,y)=\begin{bmatrix}f(x,y)\\ g(x,y)\end{bmatrix}$. Using this matrix, we can write the linear approximation to $F(x,y)$ as

$$F(x,y)\approx F(x_0,y_0)+J(x,y)\begin{bmatrix}x-x_0\\ y-y_0\end{bmatrix}.$$

Example 1

Let $F(x,y)=\begin{bmatrix}x^2y-y^3\\ 2x^3y^2+y\end{bmatrix}$. Find the Jacobi matrix, evaluate it at $(1,2)$ and find the linear approximation at that point.

Solution:

$$J(x,y)=\begin{bmatrix}2xy & x^2-3y^2\\ 6x^2y^2 & 4x^3y+1\end{bmatrix},\qquad J(1,2)=\begin{bmatrix}4 & -11\\ 24 & 9\end{bmatrix}.$$

With $F(1,2)=\begin{bmatrix}-6\\ 10\end{bmatrix}$ we get the linear approximation as

$$F(x,y)\approx\begin{bmatrix}-6\\ 10\end{bmatrix}+\begin{bmatrix}4 & -11\\ 24 & 9\end{bmatrix}\begin{bmatrix}x-1\\ y-2\end{bmatrix}=\begin{bmatrix}4x-11y+12\\ 24x-9y+32\end{bmatrix}.$$

Example 2

Let $F(x,y)=\begin{bmatrix}ye^{-x}\\ \sin(x)+\cos(y)\end{bmatrix}$. Find the Jacobi matrix, evaluate it at $(0,0)$ and find the linear approximation at that point.

Solution:

$$J(x,y)=\begin{bmatrix}-ye^{-x} & e^{-x}\\ \cos(x) & -\sin(y)\end{bmatrix},\qquad J(1,2)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

With $F(0,0)=\begin{bmatrix}0\\ 1\end{bmatrix}$ we get the linear approximation as

$$F(x,y)\approx\begin{bmatrix}0\\ 1\end{bmatrix}+\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x-0\\ y-0\end{bmatrix}=\begin{bmatrix}y\\ x+1\end{bmatrix}.$$

Example 3

Let $F(x,y) = \begin{bmatrix} \sqrt{2x+y} \\ x - y^2 \end{bmatrix}$. Find the Jacobi matrix, evaluate it at $(1,2)$ and find the linear approximation at that point.

Solution:

$$J(x,y) = \begin{bmatrix} \frac{1}{\sqrt{2x+y}} & \frac{1}{2\sqrt{2x+y}} \\ 1 & -2y \end{bmatrix}, \qquad J(1,2) = \begin{bmatrix} 1/2 & 1/4 \\ 1 & -4 \end{bmatrix}.$$

With $F(0,0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ we get the linear approximation as

$$F(x,y) \approx \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 1/2 & 1/4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix} = \begin{bmatrix} x/2 + y/4 + 1 \\ x - 4y + 4 \end{bmatrix}.$$